# SYMBOLIC DYNAMICS IN FLOWS ON THREE-MANIFOLDS

BY

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ABSTRACT. This article deals with the problem of what suspended subshifts of finite type can be realized as a basic set of a nonsingular Smale flow on three-dimensional manifolds. It is shown that any suspended subshift can be realized in such a flow on some three-manifold. Also if signs reflecting orientation are included in the matrix of the subshift of finite type then there is an obstruction to the realization on $S^3$ of basic sets corresponding to some matrices.

The qualitative study of structurally stable dynamical systems has relied heavily on the concept of symbolic dynamics (see [S]). One of the fundamental questions has been what kind of symbolic dynamics can occur on given manifolds. This sort of realization problem has been much studied and it has become clear that there are special difficulties in low dimensions—dimension 2 for diffeomorphisms and 3 for flows (see [F1, B1-F2, PS, Fr]). Many of the questions addressed by these papers were answered in higher dimensions by [Wms].

The simplest forms of symbolic dynamics occurring in flows are the suspensions of basic subshifts of finite type (see §1 for definitions). The main result of [PS] was to show that the suspension of any basic subshift of finite type can be realized as a basic set in a Smale flow on $S^3$ (or any other three-manifold). Actually their proof shows somewhat more; namely that any structure matrix can be realized. The structure matrix is the matrix of the subshift of finite type with signs added to give orientation information (see §1). The construction of Pugh and Shub, however, results in a flow with a potentially large number of singularities. The question of realizing a suspended basic subshift of finite type in a nonsingular flow on $S^3$ remains open. The present article addresses the question of realizing basic sets with a given structure matrix in nonsingular Smale flows.

THEOREM 1. *Suppose A is an abstract structure matrix for a basic set. Then there exists a nonsingular Smale flow $\phi_t$ on some three-manifold M with basic set $\Lambda$ whose structure matrix is A. Every other basic set of $\phi_t$ consists of a single closed orbit.*

Thus every structure matrix and hence every suspension of a basic subshift of finite type can be realized in a nonsingular Smale flow on some three-manifold (the manifold depends on the subshift).

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The following result shows that there is, however, an obstruction to realizing a structure matrix in a nonsingular flow on $S^3$. In particular the structure matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

cannot be realized.

THEOREM 2. *Suppose* $\phi_t$ *is a nonsingular Smale flow on* $S^3$ *with a basic set with* $n \times n$ *structure matrix* $A$. *Then if* $\det(I - A) \neq 0$ *the group* $Z^n/(I - A)Z^n$ *must be cyclic.*

If $A$ is the matrix cited above then $\det(I - A) = -4$ and it is easy to check that $Z^2/(I - A)Z^2$ is the noncyclic group of order 4. Hence this matrix cannot be realized as a structure matrix of a basic set for a nonsingular Smale flow on $S^3$.

**1. Background and definitions.** If $f_t$ is a smooth flow on a compact manifold $M$ it is said to be *structurally stable* provided that for any sufficiently close $C^1$ approximation $g_t$ there is a homeomorphism $h: M \to M$ carrying orbits of $f$ to orbits of $g$ and preserving the direction of orbits. All known examples of structurally stable flows have a hyperbolic chain-recurrent set, so we now define these concepts.

A point $x$ of $M$ is called *chain-recurrent* (see [C]) for $f_t$ provided that corresponding to any $\varepsilon$, $T > 0$ there exist points $x = x_0, x_1, \ldots, x_n = x$ and real numbers $t_0, t_1, \ldots, t_{n-1}$ all greater than $T$ such that $d(f_{t_i}(x_i), x_{i+1}) < \varepsilon$ for all $0 \leqslant i \leqslant n - 1$. The set of all such points, called the chain-recurrent set $\mathcal{R}$, is a compact set invariant under the flow.

A compact invariant set $K$ for a flow $f_t$ is said to have a *hyperbolic structure* provided that the tangent bundle of $M$ restricted to $K$ is the Whitney sum of three bundles $E^s \oplus E^u \oplus E^c$ each invariant under $Df_t$ for all $t$ and
(a) the vector field tangent to $f_t$ spans $E^c$;
(b) there are $C$, $\alpha > 0$ such that

$$\|Df_t(v)\| \leqslant Ce^{-\alpha t}\|v\| \quad \text{for } t \geqslant 0 \text{ and } v \in E^s,$$

$$\|Df_t(v)\| \geqslant Ce^{\alpha t}\|v\| \quad \text{for } t \geqslant 0 \text{ and } v \in E^u.$$

A result of Smale [S] shows that if the chain recurrent set $\mathcal{R}$ of a flow has a hyperbolic structure then $\mathcal{R}$ is the union of a finite collection of disjoint invariant compact sets each of which contains a dense orbit. Each of these sets is called a *basic set*.

Bowen [B] gave a complete characterization of one-dimensional basic sets showing that up to homeomorphism they are suspensions of basic subshifts of finite type.

1.1. DEFINITION. Given an $n \times n$ matrix $A$ of zeroes and ones, let

$$\Sigma_A = \left\{ \mathbf{x} \in \prod_{-\infty}^{\infty} \{1, \ldots, n\} \mid A_{x_k x_{k+1}} = 1 \text{ for all } k \right\}$$

and define the right shift map $\sigma: \Sigma_A \to \Sigma_A$ by $\sigma(\mathbf{x}) = \mathbf{y}$ where $y_k = x_{k-1}$. Then $\Sigma_A$ is a compact zero-dimensional space and $\sigma$ is called a *subshift of finite type*. It is called a *basic* subshift of finite type if it has a dense orbit and its periodic points are dense.

1.2. DEFINITION. *The suspension flow* of $\sigma$ is defined as follows: let $X_A = \Sigma_A \times [0, 1]/\sim$ where $\sim$ identifies $(\mathbf{x}, 1)$ with $(\sigma(\mathbf{x}), 0)$ and the flow $\phi_t$ on $X_A$ is defined by $\phi_t(\mathbf{x}, s) = (\mathbf{x}, s + t)$ for $t + s[0, 1]$ and for other $t$ by using identification.

We can alter the matrix $A$ so that it includes information on the structure of the bundles $E^u$ and $E^s$. Suppose $\Omega$ is a basic set and $h: X_A \to \Omega$ is the homeomorphism given by the result of Bowen cited above.

We consider the cross-section to the flow $f_t$ given by $h(\Sigma_A)$, where $\Sigma_A = \Sigma_A \times \{0\} \subset X_A$, and the first return map $\rho: h(\Sigma_A) \to h(\Sigma_A)$ under the flow $f_t$ ($h \mid \Sigma_A$ will be a topological conjugacy between $\sigma$ and $\rho$). The bundle $E^u$ restricted to $h(\Sigma_A)$ is trivial, since $h(\Sigma_A)$ is zero dimensional, so we can choose an orientation for it. If we let $C_i = \{\mathbf{a} \in \Sigma_A \mid a_0 = i\}$ and $A$ is chosen so the $C_i$ are sufficiently small, then the function

$$\Delta(x) = \begin{cases} 1 & \text{if } D\rho_x: E^u_x \to E^u_{\rho(x)} \text{ preserves orientation,} \\ -1 & \text{if } D\rho_x: E^u_x \to E^u_{\rho(x)} \text{ reverses orientation,} \end{cases}$$

is constant on $C_i$ and we define its value on $C_i$ to be $\Delta_i$.

1.3. DEFINITION. The matrix $B = (b_{ij})$ defined by $b_{ij} = \Delta_j a_{ij}$ will be called a *structure matrix* for the basic set.

Clearly the matrix $B$ contains all the information of $A$ and it is not difficult to see that the bundle $E^u$ is isomorphic to the bundle $\Sigma_A \times [0, 1] \times R^k/\sim$, where $k = $ fiber dim $E^u$ and $\sim$ identifies $(\mathbf{a}, 1, v)$ with $(\sigma(\mathbf{a}), 0, r(\mathbf{a}, v))$ where $r(\mathbf{a}, v) = v$ if $b_{a_1 a_0} = 1$ and $R(v)$ ($R$ an orientation reversing involution of $R^k$) if $b_{a_1 a_0} = -1$. If the ambient manifold is orientable the matrix $B$ also determines the isomorphism class of $E^s$.

1.4. THEOREM [S]. *If a flow $\phi_t$ on $M$ has a hyperbolic chain recurrent set consisting of the basic sets $\{\Omega_i\}$ then there exist manifolds with boundary $M_0 \subset M_1 \subset \cdots \subset M_n = M$ of the same dimension as $M$ which are invariant under $\phi_t$ and satisfy*

$$\bigcap_{t=-\infty}^{\infty} \phi_t(M_i - M_{i-1}) = \Omega_i.$$

The set of manifolds $M_i$ is called a *filtration* associated to the flow $\phi_t$.

We will make use of the following well-known version of the Poincaré-Hopf formula.

1.5. THEOREM. *Suppose $\phi_t$ is a nonsingular flow on an odd-dimensional manifold $M$ exiting through $N_1$ and entering on $N_2$, where $\partial M = N_1 \cup N_2$. Then $\chi(M) = \chi(N_1) = \chi(N_2)$ where $\chi(M)$ denotes the Euler characteristic.*

## 2.

2.1. PROPOSITION. *Suppose $N_1$ and $N_2$ are (not necessarily connected) surfaces with the same Euler characteristic. There exists a manifold $M^3$ with a nonsingular Morse-Smale flow $\phi$ such that $\partial M^3 = N \cup N'$, and $\phi$ exits transversely on $N$ and enters transversely on $N'$.*

PROOF. We start with the flow on $N \times I$ tangent to the $I$ factor and alter it by adding round handles as in [**As**] (see also [**F**]). Let $M_0 = M^3$ and $N_0 = N \times \{1\}$.

Suppose that $N_0$ has more than one component and some component is not $S^2$. Then we can attach a round handle $H = S^1 \times I \times I$ in such a way that $S^1 \times I \times \{0\}$ does not disconnect the component of $N_0$ which contains it and so that $S^1 \times I \times \{1\}$ lies in a different component of $N_0$. The resulting three-manifold $M_1$ will possess a nonsingular Morse-Smale flow exiting on $N \times \{0\}$ and entering on a surface $N_1$ which is $\partial M_1 - (N \times \{0\})$. The surface $N_1$ will have fewer components than $N_0$. We can repeat this process (attaching a round handle to $N_1$, etc.) as long as possible, but it finally terminates with a Morse-Smale flow $\phi$ on $M_k$ entering on $N_k$ where $N_k$ is either connected or consists entirely of spheres. In any case $N_k$ is determined up to diffeomorphism by its Euler characteristic and, hence, by the Euler characteristic of $N$ since $\chi(N) = \chi(N_k)$, by 1.5.

We now repeat the same construction (with the flow going in the opposite direction) starting with $N' \times I$. We obtain a nonsingular Morse-Smale flow on a three-manifold $M_j'$ entering on $N'$ and exiting on $N_j'$, where $N_j'$ is either connected or consists of spheres; since $\chi(N_j') = \chi(N') = \chi(N) = \chi(N_k)$ it follows that $N_j'$ is diffeomorphic to $N_k$. Gluing the flows on $M_k$ and $M_j'$ together by a diffeomorphism of $N_k$ to $N_j'$ gives the required result.   Q.E.D.

2.2. COROLLARY. *If $N$ is an oriented surface (not necessarily connected) and $\chi(N) = 0$, then there is a nonsingular Morse-Smale flow $\phi$ on a three-manifold $M$ such that $\partial M = N$ and $\phi$ exits transversely on $N$.*

PROOF. By 2.1 there is a flow on a three-manifold $M'$ exiting on $N$, entering on a torus $T^2$ and with $\partial M' = N \cup T^2$. Let $M = M' \cup_h (S^1 \times D^2)$ with $h: \partial(S^1 \times D^2) \to T^2$ a diffeomorphism; the flow on $M'$ can be extended to $M$ by placing a single repelling closed orbit in $S^1 \times D^2$. This gives the desired flow.   Q.E.D.

2.3. THEOREM 1. *Suppose $A$ is an $n \times n$ abstract structure matrix for a basic set. Then there exists a nonsingular Smale flow $\phi_t$ on some three-manifold $M$ with a basic set $\Lambda$ whose structure matrix is $A$. Every other basic set of $\phi_t$ consists of a single closed orbit.*

PROOF. A result of Pugh and Shub [**PS**] asserts the existence of a Smale flow (with singularities) on $S^3$ realizing $A$ as a structure matrix. (Actually Pugh and Shub consider only unsigned matrices but their proof is equally valid for signed structure matrices). From 1.4 it follows that there is a neighborhood $M_0$ of $\Lambda$ in $S^3$ which is a manifold satisfying:

(1) no basic set other than $\Lambda$ intersects $M_0$;

(2) the boundary $\partial M_0 = N^s \cup N^u$ with the flow entering transversely on $N^s$ and exiting transversely on $N^u$.

From 1.5 it follows that $\chi(N^s) = \chi(N^u)$. Let $N$ be a surface with $\chi(N) = -\chi(N^u) = -\chi(N^s)$ and define $M_1$ to be the disjoint union of $M_0$ and $N \times I$. We extend the flow $\phi$ to all of $M_0$ by making it tangent to the $I$ factor on $N \times I$. Note that the exit and entrance sets of $\phi$ on $M_1$ both have Euler characteristic zero. By 2.2

there is a three-manifold $P$ with $\partial P$ diffeomorphic to $N^s \cup N$ and a Morse-Smale flow on $P$ exiting on $\partial P$. Gluing $P$ to $M$ produces a Smale flow with empty entrance set. Likewise we can obtain $Q$ with a Smale flow whose entrance set is $\partial Q$ and diffeomorphic to $N^u \cup N$. Gluing via this diffeomorphism gives the desired three-manifold and flow.   Q.E.D.

**3.**

3.1. LEMMA. *If $\phi_t$ is a nonsingular flow on $S^3$ and $M \subset S^3$ is a connected embedded surface transverse to the flow then $M$ is a torus.*

PROOF. The surface $M$ is embedded, hence orientable, so it separates $S^3$. Cutting along $M$ produces a three-manifold with nonsingular flow entering transversely on $M$ and with no exit. It follows from 1.5 that the Euler characteristic of $M$ is 0. Q.E.D.

3.2. PROPOSITION. *Suppose $\psi$ is a nonsingular Smale flow on $S^3$ and $\Lambda$ is a basic set with structure matrix $A$. Then there exists a nonsingular Smale flow $\phi$ on $S^3$ with a basic set $\Omega$ which has structure matrix $A$ and with each other basic set being either an attracting closed orbit or a repelling closed orbit. If $\det(I - A) \neq 0$ then $\phi$ has only three basic sets: an attracting closed orbit, a repelling closed orbit, and $\Omega$.*

PROOF. Suppose $M_0 \subset M_1 \subset \cdots \subset M_k \subset \cdots \subset S^3$ is a filtration associated to $\psi$ (as in 1.4) and $\Lambda \subset M_k - M_{k-1}$. Let $W = \mathrm{cl}(M_k - M_{k-1})$ so $W$ is a manifold with boundary and $\psi$ is transverse to $\partial W$. Let $W = N_1 \cup \cdots \cup N_k$ where each $N_i$ is a connected component of $\partial W$. By 3.1 each $N_i$ is a torus.

Let $U$ and $V$ denote the closures of the two components of $S^3 - N_1$, so $S^3$ is formed by gluing together $U$ and $V$ along their boundaries (which are each $N_1$). Say $\Lambda \subset U$. By the torus theorem (see [**R**]), at least one of $U$ and $V$ is a solid torus $S^1 \times D^2$. Suppose first that $V$ is a solid torus. Then if $V$ contains any basic sets other than an attracting closed orbit or a repelling closed orbit, we replace it by a solid torus with a single closed orbit which is repelling if the flow exited $V$ across $\partial V$. This does not affect the flow on $W$ or change $\Lambda$ but decreases the number of basic sets which are not attractors or repellers.

If $U$ (which contains $\Lambda$) is the component of $S^3 - N_1$ which is a solid torus (perhaps knotted), we discard $V$ and form a flow on $S^3$ by gluing $U$ together with a solid torus on which there is a flow with a single closed orbit, either attracting or repelling. The solid torus $U$ will be unknotted in this $S^3$.

Repeating this process for each $N_k$ we will have removed all basic sets which are not attractors or repellers except for $\Lambda \subset W$, which we now call $\Omega$.

To prove the last assertion of the theorem we suppose that $\phi$ has been constructed as above and $\det(I - A) \neq 0$. Let $M_0 \subset \cdots \subset M_k \subset \cdots \subset S^3$ be a filtration associated to $\phi$ and suppose $\Omega \subset M_k - M_{k-1}$. If we let $X = M_k$, $A = M_{k-1}$, then $A$ consists of a finite set of solid tori each one a neighborhood of an attracting closed orbit, while $S^3 - X$ is a set of solid torus neighborhoods of repellers. By a result of [**BF**] (see also (9.11) of [**F**]) the homology $H_1(X, A; Q) = 0$ since $\det(I - A) \neq 0$. From the exact sequence $H_1(X, A; Q) \to H_0(A; Q) \to H_0(X; Q)$ and the fact that $X$

is connected, it follows that $A$ is connected. Hence there is a single attracting closed orbit. The same argument applied to the inverse flow shows there is a single repelling closed orbit.   Q.E.D.

3.3. THEOREM 2. *Suppose* $\phi_t$ *is a nonsingular Smale flow on* $S^3$ *with a basic set* $\Lambda$ *with* $n \times n$ *structure matrix* $A$. *Then if* $\det(I - A) \neq 0$ *the group* $Z^n/(I - A)Z^n$ *must be cyclic.*

PROOF. By 3.2 we may suppose that the flow $\phi$ has only three basic sets: an attractor, a repeller, and $\Lambda$. Thus a filtration associated to $\phi$ has the form $A \subset X \subset S^3$ where $A$ is a (solid torus) neighborhood of the attractor, $\Lambda \subset X - A$ and the repeller is in $S^3 - X$. By a result of [BF, Theorem 5.3], we know that

$$Z^n/(I - A)Z^n \cong H_1(X, A; Z).$$

Consider the exact sequence of the pair $(X, A)$:

$$H_1(A) \to H_1(X) \xrightarrow{i} H_1(X, A) \to H_0(A) \xrightarrow{\cong} H_0(X).$$

Since $A$ and $X$ are connected, $H_0(A) \to H_0(X)$ is an isomorphism, so the map $i$: $H_1(X) \to H_1(X, A)$ is surjective. However since $S^3 - X$ is a (solid torus) neighborhood of the repeller, $H_1(S^3 - X) = Z$. By Alexander duality $H_1(X) = Z$, so $H_1(X, A)$ is cyclic.   Q.E.D.

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